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CALCULATION OF HYDRODYNAMIC INTERACTION BETWEEN DROPS AT LOW REYNOLDS NUMBERS

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The axisymmetric problem of motion of two fluid spheres in a viscous medium is considered in the Stokes approximation. An asymptotic solution is derived for the case of a small gap between the spheres. The case when one of the spheres is solid is also considered.

The axisymmetric problem of slow motion of two spherical drops in a viscous medium was solved in [1]. In that solution, which generalizes investigations presented in [2 —5], the hydrodynamic forces are represented by infinite series. Since these series are slowly convergent, they are unsuitable for numerical computation when the gap between spheres is small. Here we derive an asymptotic solution which is also applicable in the case of fluid spheres of which one is contained inside the other. This is of interest in the study of motion of a sphere containing a gas bubble.

The obtained solution is substantially different from the asymptotic solution for solid spheres [6].

1. Statement of the problem. Let us consider two fluid spheres of radius a and b moving at velocities V_a and V_b, respectively. Two possible relative positions of spheres are shown in Fig. 1. The Reynolds number and the relative velocities of the spheres are assumed small, and the problem is analyzed in the Stokes approximation.

Impermeability of contact surfaces to the fluid, and the continuity of velocity and tangent stress at the sphere surfaces are taken as the boundary conditions. We assume the surface tension at interfaces of liquids [in the sphere and outside it] to be fairly high, so that it is possible to neglect any deviation of the drop shape from the spherical, and omit the consideration of continuity of normal stresses.

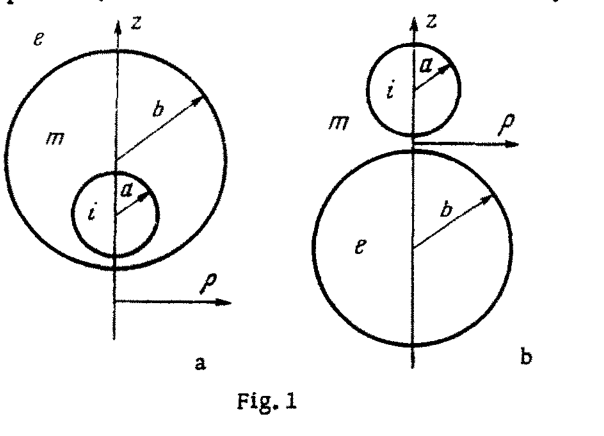

Fig. 1

Owing to the problem linearity the force acting on a sphere of radius a can in the Stokes approximation be represented as

$$F_a = -6\pi\mu_m a\,[\Lambda_{11}(V_a - V_b) + \Lambda_{12}V_b]$$

When sphere a is located inside sphere b (see Fig. 1, a), the force acting on the external sphere is of the form

$$F_b = -6\pi\mu_e b\,[\Lambda_{21}(V_a - V_b) + \Lambda_{22}V_b]$$

where μ_m and μ_e are the viscosities of fluids in regions m and e (see Fig. 1), and subscripts m, e, and i denote quantities in regions shown in that figure.

Using the reciprocity theorem [7] and taking into account boundary conditions it is possible to show that

$$\Lambda_{21} = ab^{-1}\mu_m\mu_e^{-1}\Lambda_{12} \tag{1.1}$$

When the spheres are tangent the quantities Λ_{12}, Λ_{22}, and Λ_{21} remain finite, which for Λ_{21} follows form (1.1). The limit value of Λ_{12} was determined in [1, 8, 9] for solid and fluid spheres in case b.

The aim of the present investigation is to determine the asymptotics of Λ_{11} when the gap between the sphere surfaces is small.

2. **Solution.** Using the bispherical system of coordinates

$$z = \frac{c \operatorname{sh} \eta}{\operatorname{ch} \eta - \mu}, \quad \rho = \frac{c \sin \xi}{\operatorname{ch} \eta - \mu}, \quad \mu = \cos \xi \tag{2.1}$$

it is possible to have the sphere of radius a to be the coordinate surface $\eta =$ const $= \eta_1 > 0$ and the sphere of radius b to represent the coordinate surface $\eta =$ const $= \eta_0$. The quantities c, η_1, and η_0 are determined by formulas

$$\operatorname{ch} \eta_1 = \frac{(1+\varepsilon)(1-k) - k\varepsilon^2/2}{1-k-k\varepsilon}, \quad \operatorname{sh} \eta_0 = k \operatorname{sh} \eta_1, \quad c = a \operatorname{sh} \eta_1 \tag{2.2}$$

where εa is the gap between spheres, $k = \pm a/b$, and the minus sign relates to the position of spheres shown in Fig. 1, b.

The stream function was obtained in [1] in the case of $V_b = 0$ for all flow regions in the form

$$\Psi = 2^{-1/2}c^2V_a(\operatorname{ch}\eta - \mu)^{-3/2}\sum_{n=1}^{\infty} n(n+1)\varphi_n(\eta)Q_n(\mu) \tag{2.3}$$

$$\varphi_n^m(\eta) = E_n \operatorname{ch}(n - 1/2)\eta + F_n \operatorname{sh}(n - 1/2)\eta + G_n \operatorname{ch}(n + 3/2)\eta + H_n \operatorname{sh}(n + 3/2)\eta$$

$$\varphi_n^e(\eta) = C_n \exp(n - 1/2)\eta + D_n \exp(n + 3/2)\eta$$
$$\varphi_n^i(\eta) = A_n \exp[-(n - 1/2)\eta)] + B_n \exp[-(n + 3/2)\eta]$$

where V_a is the projection of velocity $\mathbf{V}_a$ on the negative axis z and $Q_n(\mu)$ is the Gegenbauer polynomial related to the Legendre polynomials by formula

$$Q_n(\mu) = [P_{n+1}(\mu) - P_{n-1}(\mu)]/(2n+1)$$

Investigations in [1] had shown that the boundary conditions assume the form

$$\eta = \eta_0, \quad \varphi_n^m = \varphi_n^e = 0, \quad d\varphi_n^m/d\eta = d\varphi_n^e/d\eta \tag{2.4}$$
$$d^2\varphi_n^m/d\eta^2 = \lambda_e d^2\varphi_n^e/d\eta^2, \quad \lambda_e = \mu_e/\mu_m$$
$$\eta = \eta_1, \ \varphi_n^m = \varphi_n^i = R_n, \quad d\varphi_n^m/d\eta = d\varphi_n^i/d\eta$$
$$d^2\varphi_n^m/d\eta^2 - \lambda_i d^2\varphi_n^i/d\eta^2 = (1-\lambda_i)\, d^2R_n/d\eta^2, \quad \lambda_i = \mu_i/\mu_m$$

$$R_n(\eta) = \frac{\exp[-(n+3/2)\eta]}{2n+3} - \frac{\exp[-(n-1/2)\eta]}{2n-1} \tag{2.5}$$

When $V_b = 0$ formulas (2.3)—(2.5) are also valid in case a, which justifies the use of a common designation of flow regions in cases a and b and makes possible the investigation of asymptotics of Λ_{11} for any relative position of spheres.

Let us determine the inner expansion for the stream function Ψ^m, which is valid in the region of small gap between the sphere surfaces, since that region determines the singular part of Λ_{11}. Using (2.5) and the last two of formulas (2.3) we eliminate from equalities (2.4) functions $\varphi_n^i(\eta)$ and $\varphi_n^e(\eta)$ and obtain for $\varphi_n^m(\eta)$ the closed system of boundary conditions

$$\begin{aligned}&\eta = \eta_0, \quad \varphi_n^m = 0, \; d^2\varphi_n^m / d\eta^2 = \lambda_e (2n+1)\, d\varphi_n^m / d\eta \\ &\eta = \eta_1, \quad \varphi_n^m = R_n, \quad d^2\varphi_n^m / d\eta^2 + \lambda_i (2n+1)\, d\varphi_n^m / d\eta = \\ &\quad d^2R_n / d\eta^2 + \lambda_i (2n+1)\, dR_n / d\eta\end{aligned} \tag{2.6}$$

The tangency case corresponds to the passing to limit

$$\eta_0 \to 0, \; \eta_1 \to 0 \tag{2.7}$$

where $\operatorname{sh} \eta_0 / \operatorname{sh} \eta_1 = k$ is fixed. It follows from (2.7) that $\eta_0/\eta_1 = k + k(1 - k^2)\,\eta_1^2 / 6 + O(\eta_1^4)$. We introduce the variable $\sigma = \eta / \eta_1$ and note that the inner region corresponds to $\sigma \sim 1$ and $1 - \mu \sim 1$. We fix the values of σ, n, λ_i, and λ_e, pass to limit by applying (2.7) to (2.5) and (2.6), and to the differential equation for $\varphi_n^m(\eta)$ implied by the second of equalities (2.3), and obtain

$$\begin{aligned}&\varphi_n^m(\eta) = \alpha_n(\sigma) + \eta_1\beta_n(\sigma) + \eta_1^2\gamma_n(\sigma) + O(\eta_1^3) \\ &\alpha_n(\sigma) = -\frac{4(\sigma - k)}{(2n-1)(2n+3)(1-k)}, \\ &\beta_n(\sigma) = \frac{(2n+1)(A_1\sigma^3 + B_1\sigma^2 + C_1\sigma + D_1)}{(2n-1)(2n+3)} \\ &\gamma_n(\sigma) = \frac{2k(1+k)(1-\sigma)}{3(1-k)(2n-1)(2n+3)} + A_2\sigma^3 + B_2\sigma^2 + C_2\sigma + D_2 + \\ &\quad \frac{(2n+1)^2(A_3\sigma^3 + B_3\sigma^2 + C_3\sigma + D_3)}{(2n-1)(2n+3)}\end{aligned} \tag{2.8}$$

where A_j, B_j, C_j, and D_j $(j = 1, 2, 3)$ depend on λ_i, λ_e, and k. The singular part of Λ_{11} is determined by coefficients A_1, A_2, and A_3

$$A_1 = \frac{2(\lambda_i + \lambda_e)}{3(1-k)^2}, \quad A_2 = \frac{1}{6(1-k)}, \quad A_3 = \frac{2(\lambda_i\lambda_e - \lambda_i^2 - \lambda_e^2)}{9(1-k)} \tag{2.9}$$

The first two terms of the inner expansion of Ψ^m can be directly obtained by substituting (2.8) into the series (2.3). The third term is determined by, first, summating series (2.3) by parts and then expressing the Gegenbauer polynomials in terms of Legendre polynomials. As the result, we have

$$\begin{aligned}&\Psi^m \simeq V_a c^2 [\Psi_0(\sigma, \mu) + \eta_1\Psi_1(\sigma, \mu) + \eta_1^2\Psi_2(\sigma, \mu) + O(\eta_1^3)] \\ &\Psi_0 = \frac{(\sigma - k)(1+\mu)}{2(1-k)(1-\mu)}, \quad \Psi_1 = 2^{-1/2}(A_1\sigma^3 + B_1\sigma^2 + C_1\sigma + D_1)\,T(\mu) \\ &\Psi_2 = \frac{k(1+k)(\sigma - 1)(1+\mu)}{12(1-k)(1-\mu)} - \frac{(A_2\sigma^3 + B_2\sigma^2 + C_2\sigma + D_2)(1+\mu)}{4(1-\mu)^2} + \\ &\quad \frac{(A_3\sigma^3 + B_3\sigma^2 + C_3\sigma + D_3)(1+\mu)(2\mu - 3)}{4(1-\mu)^2} - \frac{3\sigma^2(\sigma - k)(1+\mu)}{8(1-\mu)^2(1-k)}\end{aligned} \tag{2.10}$$

$$T(\mu) = 6(1-\mu)^{-3/2}\sum_{n=0}^{\infty}\frac{P_n(\mu)}{(2n-3)(2n+1)(2n+5)}$$

The last of these formulas implies that when $\mu \to +1$

$$T(\mu) = -\tfrac{1}{2}(1-\mu)^{-3/2} + O([1-\mu)^{-1/2}\ln(1-\mu)] \tag{2.11}$$

It will be seen from (2.10) and (2.11) that the validity of the inner expansion is violated when $1-\mu \sim \eta_1^2$. We introduce the variable $\tau = \eta_1^{-1}\sqrt{2(1-\mu)}$, noting that in the first approximation σ and τ are tangent—spherical coordinates

$$z / a = 2\sigma / (\sigma^2 + \tau^2), \quad \rho / a = 2\tau / (\sigma^2 + \tau^2)$$

Using the general solution of the Stokes equation for stream function in coordinates σ, τ derived in [6, 8] and satisfying boundary conditions, it is possible to formulate the first term of the external expansion of Ψ^m, and then using (2.11) to show that the external expansion merges with the internal.

The force acting on a sphere of radius a is obtained, in conformity with [2], in the form

$$F_a = \pi\mu_m \int \rho^3 \frac{\partial}{\partial n}\left(\frac{E^2\Psi^m}{\rho^2}\right) ds \tag{2.12}$$

where integration is carried out along the arc of the half of the sphere meridian cross section, n is the direction of the outer normal, and the Stokes operator E^2 in bispherical coordinates is

$$E^2 = \frac{(\operatorname{ch}\eta-\mu)}{c^2}\left[\frac{\partial}{\partial\eta}\left((\operatorname{ch}\eta-\mu)\frac{\partial}{\partial\eta}\right) + (1-\mu^2)\frac{\partial}{\partial\mu}\left((\operatorname{ch}\eta-\mu)\frac{\partial}{\partial\mu}\right)\right] \tag{2.13}$$

It follows from (2.10)—(2.13) that the contribution of the inner region to coefficient Λ_{11} is

$$-\frac{1}{6}\int_{-1}^{\mu_0} d\mu\left[\eta_1^{-1}(1-\mu)\frac{\partial^3\Psi_1}{\partial\sigma^3} + \frac{\partial\Psi_0}{\partial\sigma} + (1-\mu)\frac{\partial^3\Psi_2}{\partial\sigma^3} + (1-\mu^2)\frac{\partial}{\partial\mu}\left((1-\mu)\frac{\partial^2\Psi_0}{\partial\sigma\partial\mu}\right)\right] \tag{2.14}$$

The value of μ_0 lies in the region of overlap of the external and inner expansions. Terms which tend to zero when $\eta_1 \to 0$ are omitted. From (2.10) we have

$$\int_{-1}^{1}(1-\mu)T(\mu)\,d\mu = -\frac{3\pi^2\sqrt{2}}{32} \tag{2.15}$$

Using (2.10), (2.11), and (2.15) we represent (2.14) in the form

$$3/32\pi^2 A_1\eta_1^{-1} = [{}^1\!/_4(1-k)^{-1} + {}^1\!/_2(A_2 + A_3)]\,[\ln(\eta_1^2/2) + \ln(\tau_0^2)] -$$
$$A_1\tau_0/2 + O[\eta_1^{-1}(1-\mu_0)^{3/2}\ln(1-\mu_0)] + O(1)$$

It should be noted that the terms containing τ_0 cancel out in the contribution of the external region to coefficient Λ_{11} and that besides these terms the contribution of the x external region is only of order $O(1)$.

The final result, with allowance for (2.2) and (2.9), is of the form

$$\Lambda_{11} = (^1/_{32})\,\pi^2 \sqrt{2}\,(1-k)^{-3/2}(\lambda_i + \lambda_e)\,\varepsilon^{-1/2} - {}^1/_3\,(1-k)^{-1}\,[1 + (\lambda_i\lambda_e - \lambda_i^2 - \lambda_e^2)\,/\,3]\ln\varepsilon + O(1) \tag{2.16}$$

Formula (2.16) was obtained on the assumption that $\varepsilon \to 0$ when λ_i and λ_e have fixed finite values, and it shows that the region of the derived solution applicability is bounded by the condition

$$\max(\lambda_i,\ \lambda_e) \ll (\sqrt{\varepsilon}\,|\ln\varepsilon|)^{-1} \tag{2.17}$$

which makes it impossible to pass from (2.16) to the asymptotic formula for solid bodies [6].

The quantity Λ_{11} was determined in [1] by the sum of a slowly convergent series for small ε. Results of numerical calculations by formulas in [1] for $k = {}^1/_2$ are compared below with the values of Λ^*_{11} determined by the asymptotic formulas (2.16)

$\lambda_i = \lambda_e$	ε	Λ_{11}	$(\Lambda^*_{11} - \Lambda_{11})/\Lambda_{11}$
$\sqrt{3}$	$2\cdot 10^{-2}$	$2.565\cdot 10$	$1.8\cdot 10^{-1}$
$\sqrt{3}$	$2\cdot 10^{-4}$	$2.967\cdot 10^{2}$	$1.8\cdot 10^{-2}$
$\sqrt{3}$	$2\cdot 10^{-6}$	$3.016\cdot 10^{3}$	$1.9\cdot 10^{-3}$
3	$2\cdot 10^{-2}$	$3.890\cdot 10$	$2.1\cdot 10^{-1}$
3	$2\cdot 10^{-4}$	$5.019\cdot 10^{2}$	$2.0\cdot 10^{-2}$
3	$2\cdot 10^{-6}$	$5.206\cdot 10^{3}$	$2.0\cdot 10^{-3}$

The described here method can be used for investigating the case when one of the spheres is solid. Function $\alpha_n(\sigma)$ defined in (2.8) is then a third power polynomial, hence it is necessary to add to (2.14) the expression

$$-\frac{1}{6}\int_{-1}^{\mu_0} d\mu \left[\frac{(1-\mu)}{\eta_1^2}\frac{\partial^3\Psi_0}{\partial\sigma^3} + 5\frac{\partial^2\Psi_0}{\partial\sigma^2} + \frac{1}{2}\frac{\partial^3\Psi_0}{\partial\sigma^3}\right] \tag{2.18}$$

All derivatives in (2.18) and (2.14) are determined for $\sigma = 1$. Otherwise this case does not greatly differ from the one considered earlier. When $\lambda_e = \infty$ we have

$$\Lambda_{11} = \frac{1}{4(1-k)^2\varepsilon} + \frac{9\pi^2\sqrt{2}}{128}(1-k)^{-3/2}\lambda_i\varepsilon^{-1/2} + \left[\frac{3\lambda_i^2}{16(1-k)} - \frac{9k^2 - 18k + 4}{20(1-k)^3}\right]\ln\varepsilon + O(1) \tag{2.19}$$

If $1 - k \sim 1$ is assumed, then, as implied by (2.19), the derived solution is valid for $\lambda_i \ll (\sqrt{\varepsilon}\,|\ln\varepsilon|)^{-1}$.

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